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Analysis of Stress Due to Fastener Tolerance in Assembled Components

Joshua H. Gordis*
Naval Postgraduate School, Monterey, California 93943
and
William G. Flannelly†
Kaman Aerospace Corporation, Bloomfield, Connecticut 06002

The location of fasteners in a manufactured component are commonly specified with an allowed deviation from nominal location, known as tolerance. The assembly of such components can generate stress due to the accumulation of these tolerances. A highly efficient and exact method for the linear static and complex dynamic analysis of assembly stress is presented. An exact reduced representation of the assembled components is generated using frequency domain structural synthesis. An alternative coordinate system is employed in the synthesis that allows the direct application of fastener tolerances to the component assembly model and the rapid calculation of the resulting displacements, strains, and stresses. The method provides an efficient means of rapidly assessing the effects of proposed (maximum allowable) tolerance limits for each component, thereby aiding in the design process and minimizing manufacturing costs.

Nomenclature

C = damping matrix
f = generalized excitation
K = stiffness matrix
M = mass matrix, mapping matrix
q = generalized response
R = equilibrium matrix or position vector
Y = frequency response function or flexibility matrix
Z = impedance matrix
ω = frequency, s^{-1}
ε = strain
σ = stress

I. Introduction

THE manufacture of components for subsequent assembly generally requires the allowance for deviations from nominal location for fasteners, commonly known as tolerance. The assembly of such components can generate stresses due to the accumulation of these tolerances, or mismatch in fastener locations, which are specified for each of the assembled components. To maximize the rate and to minimize the cost of manufacturing each component, it is desirable to ascertain the maximum tolerance that can be specified for each fastener and for all components. The method to be described provides an exact and highly efficient means for calculating the displacements, strains, and stresses in the assembled structure, due to the accumulation of the tolerances. Frequency domain structural synthesis is used to analytically assemble reduced-order component models, and the specified tolerances are applied to the synthesized model of the assembled components as imposed differential displacements. The resulting displacements, strains, and stresses are directly calculated. The analogy of the mathematical statement of this problem with that of thermal and interference stress allows an equally effective treatment of these problems as well.

The methodology is cast in the frequency domain; the structural models involved are frequency response function models, also known as mobility models. As we will show, the frequency domain formulation offers both computational efficiency and an exact solution in problems of structural synthesis and the subsequent calculation of assembly strain and stress. Note that a frequency response function at zero frequency is the flexibility, hence the treatment of static problems. In the course of our discussion, the term frequency response will include informally the zero-frequency case; when required, specific reference to either (dynamic) frequency response or flexibility will be made.

The computational efficiency offered by the method is due to a large extent from the exact and dimensionally unlimited model reduction available from frequency response function models. Given a frequency response function matrix of an order equal to that of its originating finite element or test model, the analyst is free to extract from this matrix those elements corresponding to model response coordinates of interest. As we will see, these coordinates will include those directly involved in the synthesis, such as substructure connection coordinates, and additionally may include coordinates for which synthesized response information is of interest, such as points in the model where stress information is required. As already indicated, frequency response can be derived either analytically or experimentally. Therefore, the method accommodates structural models, such as finite element models, or experimental models, such as those derived from static or dynamic measurement. The generality of the definition of frequency response allows displacement, strain, and stress coordinates to be included, as desired. It is these varied and useful characteristics that make the method an ideal design tool.

II. Outline of Method

The application of the method in the analysis of assembly stress is outlined here. We seek an analytical model of the assembly of all components to which specified tolerances can be conveniently applied and resulting displacements, strains, and stresses directly calculated. To this end, the general procedure is as follows.

Step 1: A frequency response function model is generated independently for all components. For unrestrained component models (models with rigid-body modes) involved in a static analysis, an appropriate number of springs to ground are installed before the calculation of flexibilities. The minimum number of springs required equals the number of rigid-body modes of the component model.

Step 2: The component models are assembled using frequency domain structural synthesis, at the model response coordinates corresponding to the fastener locations. If grounding springs are present in an unrestrained model (in a static analysis), these springs are removed simultaneously with the component coupling. What results

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*Assistant Professor, Department of Mechanical Engineering.

†Chief Research Engineer.

is a frequency response model of the coupled system, referred to as the synthesized assembly model.

Step 3: The tolerances for the deviation from nominal location of the fasteners for each component are appropriately combined and applied to the synthesized assembly model. Displacement, strain, and stress can be directly calculated from the synthesized frequency response model, depending on whether the corresponding frequency response quantities were included in the synthesis.

The theory involved in assembly stress analysis is now presented. We begin with the background of frequency domain structural models. We define the matrix partitioning that is central to the application of the theory and follow with the theory of frequency domain structural synthesis, tailored to our problem of the analysis of assembly stress. As already outlined, the synthesis serves two purposes: The first purpose is to assemble the individual component models into a single system model. The second purpose is to remove grounding springs from originally unrestrained models, a necessity only in static analysis. In the static synthesis of several substructures, all of which are originally unrestrained, a sufficient number of grounding springs must be retained in the synthesized assembly model to preclude an attempt at the synthesis of a singular system, i.e., all grounding springs except for a number equal to the number of rigid-body modes of the synthesized model can be removed. If an insufficient number of grounding springs are retained in a static synthesized assembly model, the synthesis will fail due to its functional equivalence to the inversion of a singular matrix.

Given the aforementioned two purposes served by the synthesis, we can classify, at this point qualitatively, the various model response coordinates involved in the synthesis. Those displacement response coordinates (fastener locations) from all substructures (components) involved in the coupling are referred to as connection response coordinates. In the case of a static analysis involving one or more unrestrained component models, those displacement response coordinates of the component models at which grounding springs have been installed (before the calculation of component flexibility models, of course) are referred to as grounding response coordinates. Those displacement response coordinates not directly involved in either the coupling or the grounding, but about which we desire synthesized response information, are referred to as internal response coordinates. In addition to these classes of coordinates, strain and stress response coordinates can also be included if postsynthesis strain and stress information is required. Bold faced symbols denote vector quantities.

III. Theory

A. Frequency Domain Structural Models

A frequency response function structural model is indicated in general as

$$\boldsymbol{q}(\omega) = \boldsymbol{Y}(\omega)\boldsymbol{f}(\omega) \tag{1}$$

where $\boldsymbol{q}$ and $\boldsymbol{f}$ are vectors of complex-valued generalized response and excitation coordinates, respectively, at a specific frequency ω, and $\boldsymbol{Y}$ is an appropriately sized frequency response function matrix, evaluated at the frequency ω. In general, an element of the frequency response function matrix is defined as

$$Y_{ij} = \partial q_i / \partial f_j \tag{2}$$

the partial derivative of the ith generalized response coordinate with respect to the jth generalized excitation coordinate. The displacement-force frequency response at a frequency ω can be found from the impedance matrix $\boldsymbol{Z}$,

$$\boldsymbol{Y}(\omega) = \boldsymbol{Z}(\omega)^{-1}$$

where

$$\boldsymbol{Z}(\omega) = \boldsymbol{K} - \omega^2\boldsymbol{M} + j\omega\boldsymbol{C}$$

and where $\boldsymbol{K}$, $\boldsymbol{M}$, and $\boldsymbol{C}$ result from the finite element assembly process, and $j = \sqrt{-1}$. The flexibility, which is the frequency response evaluated at zero frequency, is found analytically from a nonsingular stiffness matrix,

$$\boldsymbol{Y}(\omega = 0) = \boldsymbol{K}^{-1}$$

Various types of frequency response can be defined depending upon the type of coordinates involved. For example, strain-force and stress-force frequency responses are defined as

$$Y_{ij}^{\varepsilon} = \partial \varepsilon_i / \partial f_j \tag{2a}$$

$$Y_{ij}^{\sigma} = \partial \sigma_i / \partial f_j \tag{2b}$$

where ε_i and σ_i are complex-valued strains and stresses, respectively, at coordinate i, at a specific frequency ω. Other frequency response functions will be defined as required.

B. Matrix Partitioning

We specialize Eq. (1) for our purpose by the following partitioning:

$$\begin{Bmatrix} \boldsymbol{q}_\sigma \\ \boldsymbol{q}_i \\ \boldsymbol{q}_c \\ \boldsymbol{q}_g \end{Bmatrix} \begin{bmatrix} \boldsymbol{Y}_{\sigma i} & \boldsymbol{Y}_{\sigma c} & \boldsymbol{Y}_{\sigma g} \\ \boldsymbol{Y}_{ii} & \boldsymbol{Y}_{ic} & \boldsymbol{Y}_{ig} \\ \boldsymbol{Y}_{ci} & \boldsymbol{Y}_{cc} & \boldsymbol{Y}_{cg} \\ \boldsymbol{Y}_{gi} & \boldsymbol{Y}_{gc} & \boldsymbol{Y}_{gg} \end{bmatrix} \begin{Bmatrix} \boldsymbol{f}_i \\ \boldsymbol{f}_c \\ \boldsymbol{f}_g \end{Bmatrix} \tag{3}$$

Equation (3), similarly to Eq. (1), is a completely general description of an arbitrary (linear) baseline structural system, here considered to be comprised of two or more structures. The subscripting in Eq. (3) defines sets of component model response coordinates corresponding to the qualitative classification of coordinates put forth earlier. The classification is now repeated, here quantitatively with respect to the role played by the various coordinates in Eq. (3). The various quantities are defined as follows:

$\boldsymbol{q}_\sigma$: A set of stress response coordinates. These coordinates are not directly involved in the coupling of substructures or the removal of grounding springs. We may also include strain response coordinates, $\boldsymbol{q}_\varepsilon$, if required.

$\boldsymbol{q}_i$,$\boldsymbol{f}_i$: A set of displacement responses and generalized excitations, respectively, at internal response coordinates not directly involved in the coupling of substructures or the removal of grounding springs.

$\boldsymbol{q}_c$,$\boldsymbol{f}_c$: A set of generalized responses and excitations, respectively, at connection response coordinates directly involved in the coupling of substructures (the joining of components at fastener locations).

$\boldsymbol{q}_g$, $\boldsymbol{f}_g$: A set of generalized responses and excitations, respectively, at grounding response coordinates directly involved in the removal of grounding springs (required for unrestrained static component models).

$\boldsymbol{Y}_{\sigma i}$, $\boldsymbol{Y}_{\sigma c}$, etc.: Frequency response function matrices defined by the previous defined sets.

From the previous classifications, each subscript in Eq. (3) refers to a formal set of model coordinates and also specifies the type of response (e.g., displacement, stress, etc.) for the coordinates in the set. Roman subscripts (i, c, and g) refer to displacement response; the Greek subscript σ indicates stress responses calculated at the model coordinates in set "σ." Note that the previously defined sets of model coordinates need not be mutually exclusive. For example, we may require stress information at a coordinate that is simultaneously a connection coordinate and a grounding coordinate. This coordinate would therefore be included in sets σ, c, and g.

In general, the connection coordinates may experience both externally applied forces and coupling forces (to be established through synthesis), i.e.,

$$\boldsymbol{f}_c = \boldsymbol{f}_c^{\text{ext}} + \boldsymbol{f}_c^{\text{cpl}} \tag{4a}$$

The grounding coordinates may experience both externally applied forces and forces due to the presence of grounding springs, i.e.,

$$\boldsymbol{f}_g = \boldsymbol{f}_g^{\text{ext}} + \boldsymbol{f}_g^{\text{cpl}} \tag{4b}$$

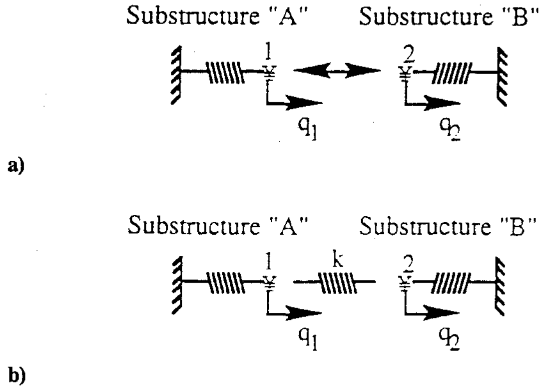


Fig. 1 a) Direct coupling and b) indirect coupling.

By definition of the subscript "i," we may have only

$$f_i = f_i^{\text{ext}} \tag{4c}$$

We define the set union $e = i \cup c \cup g$, and expand Eq. (3) as follows:

$$\begin{Bmatrix} q_\sigma \\ q_e \\ q_c \\ q_g \end{Bmatrix} = \begin{bmatrix} Y_{\sigma e} & Y_{\sigma c} & Y_{\sigma g} \\ Y_{ee} & Y_{ec} & Y_{eg} \\ Y_{ce} & Y_{cc} & Y_{cg} \\ Y_{ge} & Y_{gc} & Y_{gg} \end{bmatrix} \begin{Bmatrix} f_e \\ f_c \\ f_g \end{Bmatrix} \quad \text{where } \{f_e\} = \begin{Bmatrix} f_i^{\text{ext}} \\ f_c^{\text{ext}} \\ f_g^{\text{ext}} \end{Bmatrix} \tag{5}$$

Note the presence of redundant equations in Eq. (5).

C. Coupling Force/Differential Response Coordinates

The establishment of coupling force/differential response coordinates (to be referred to as "coupling coordinates") is a precursor to the construction of a structural synthesis transformation (SST). As will be demonstrated, the utility of the coupling coordinates derives from their intrinsic relation to the mechanics of synthesis[1,2] and from the organization they bring to the practical implementation of the theory.[3,4] Furthermore, the differential response coordinates play the critical role in making possible the direct application of the fastener tolerances to the synthesized assembly model for the calculation of system response. Note that these coordinates are not a prerequisite for structural synthesis; in fact, the development of the SST that follows can be cast without the coupling coordinates.[1,2]

The SST will operate on Eq. (5) and will affect both the coupling of component models and the removal of grounding springs. What results from this transformation of Eq. (5) is its postsynthesis analog: a frequency response function model of the synthesized assembly. Therefore, preceding the construction of the SST is the construction of an auxiliary transformation, integral to the SST, which defines the coupling coordinates. This auxiliary transformation operates on the coupling forces and the connection and grounding response coordinates and produces the corresponding coupling coordinates we desire.

We now consider the mechanics of substructure coupling and, by way of example, will refer to Figs. 1a and 1b in the course of our development. Figures 1a and 1b show a one-dimensional static direct coupling and a one-dimensional static indirect coupling, respectively. Regardless of the number of couplings or grounding spring removals involved in a synthesis, each one individually is directly analogous to those depicted in Figs. 1a and 1b, respectively. We will therefore refer to Fig. 1a in the development of the necessary relations for direct coupling (fastener connection); the example (Fig. 1b) of indirect coupling will lead to the necessary relations for the removal of a grounding spring.

We seek a change of basis from the original (connection and grounding) force and response coordinates of Eq. (5) to the coupling coordinates. We therefore require linear transformations for both the coupling forces and the connection response coordinates. The direct and indirect coupling of response coordinates involves a common statement of equilibrium; the transformation of the original forces, [Eqs. (4a) and (4b)] into the corresponding coupling forces that satisfy this equilibrium will derive from this statement. Having so constructed the transformation of forces, the corresponding transformation of response coordinates to their coupling counterparts is then identified. The development of the relations for direct coupling parallels that for indirect coupling. We conclude the section by specializing the results obtained for indirect coupling to the removal of a grounding spring.

In both cases shown in Figs. 1a and 1b, we wish to couple the two substructures "A" and "B" using their respective connection response coordinates. These coordinates taken together comprise the connection response coordinate set $\{q_c\}$. For example, the connection response coordinates associated with substructures "A" and "B" are q_1 and q_2, respectively. The connection response coordinate set is therefore

$$\{qc\} = \lfloor q_c^A \; q_c^B \rfloor^T = \lfloor q_1 \; q_2 \rfloor^T$$

Consider a pair of connection response coordinates to be coupled, either directly or indirectly, such as q_1 and q_2 in Figs. 1a and 1b. We write equilibrium for the pair in the coupled state as

$$R_c f_c = 0 \tag{6}$$

where $f_c = \lfloor f_1 f_2 \rfloor^T$ and R_c is a matrix containing coefficients appropriate for expressing equilibrium. For the indirect coupling we note that the coupling forces f_c depend on the interconnection impedance and the connection response coordinates,

$$f_c = Zq_c \tag{7}$$

where $q_c = \lfloor q_1 \; q_2 \rfloor^T$. Equation (6) can be manipulated into the form

$$f_c = M_c \tilde{f}_c \tag{8}$$

where the circumflex indicates a coupling coordinate. Equation (8) represents the transformation of forces required and is simply another statement of equilibrium. The vector $\tilde{f}_c$ is the coupling force and

$$M_c = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{9}$$

For a simple spring as an interconnection impedance,

$$Z = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$$

where k = spring rate. To identify the corresponding transformation of the connection response coordinates, we consider the work associated with the connection in the original coordinate system and in the coupling coordinate system. In the original coordinate system,

$$W = \tfrac{1}{2} f_c^T q_c = \tfrac{1}{2} \tilde{f}^T M_c^T q_c$$

and in the coupling coordinate system,

$$W_{\text{cpl}} = \tfrac{1}{2} \tilde{f}^T \tilde{q}_c = \tfrac{1}{2} \tilde{f}^T T_c q_c$$

Requiring that $W = W_{\text{cpl}}$, we find that the unknown transformation of response coordinates is $T_c = M_c^T$. Our transformation is complete:

$$f_c = M_c \tilde{f}_c \tag{10a}$$

$$\tilde{q}_c = M_c^T q_c \tag{10b}$$

The vector $\tilde{q}_c$ in Eq. (10a) contains the differential response coordinates. For a one-dimensional coupling, $\tilde{q}_c$ is a scalar where $\tilde{q}_c = q_1 - q_2$.

For a direct coupling, compatibility is enforced as

$$\tilde{q}_c = M_c^T q_c = \mathbf{0} \tag{11}$$

where Eq. (11) requires that "the two coordinates respond as one."

For an indirect coupling, it remains to identify the form of the interconnection impedance that is consistent with the coupling coordinates. We necessarily restrict our attention to interconnection impedances with no inertia.[2] Equation (7) describes such an impedance and is repeated here:

$$f_c = Zq_c$$

The coupling impedance is found from Eq. (7) as

$$\tilde{f}_c = (M_c)^\dagger Z \left(M_c^T\right)^\dagger \tilde{q}_c \tag{12}$$

and the associated coupling impedance is therefore

$$\tilde{Z} = (M_c)^\dagger Z \left(M_c^T\right)^\dagger \tag{13}$$

where $(\bullet)^\dagger$ indicates the pseudoinverse. This Eq. (12) is exact, as shown in Ref. 2, and guarantees that

$$\tfrac{1}{2} q_c^T Z q_c = \tfrac{1}{2} \tilde{q}_c^T \tilde{Z} \tilde{q}_c$$

Note that for a simple spring $\tilde{Z} = k$, $\{f\} = [Z]\{q\}$, and

$$\{f\} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

$$= k \begin{Bmatrix} (q_1 - q_2) \\ (q_2 - q_1) \end{Bmatrix} = \tilde{Z} \begin{Bmatrix} \tilde{q}_c \\ -\tilde{q}_c \end{Bmatrix} \tag{14}$$

a result consistent with that of Eq. (12).

The removal of a grounding spring is a synthesis operation that involves a single (grounding) response coordinate and is a variant of the removal (or installation) of a discrete interconnection impedance between two response coordinates. We specialize our previous results to the removal of a grounding spring from a single grounding coordinate q_g. This is achieved by fully constraining either q_1 or q_2 in Fig. 1b. We therefore delete the corresponding row in Eqs. (10a) and (10b) to obtain the relations applicable to the removal of a grounding spring. If we arbitrarily delete the second row,

$$f_g = M_g \tilde{f}_g \tag{15a}$$

$$\tilde{q}_g = M_g^T q_g \tag{15b}$$

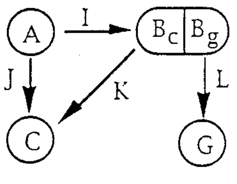
where, for a simple spring,

$$M_g = [1] \tag{16}$$

D. Boolean Representation of Connectivity: M_c and M_g

We mentioned earlier that a primary benefit of the coupling coordinates is the organization they bring to the practical implementation of the synthesis theory.[3,4] This organization is provided by the boolean transformation matrices M_c and M_g, generally denoted as M. Boolean transformation matrices are well suited to the accommodation of information pertaining to connectivity, i.e., what is connected to what, and this information conveniently corresponds to the organization of the signs of the coupling forces and reactions generated in a synthesis.[1,2] This correspondence may not be a surprise, given the origin of the boolean matrices in equilibrium. The construction of the M matrices required for a synthesis therefore serves as an "automated" way of establishing and computationally handling a sign convention for the coupling forces and reactions, hence the inclusion of the coupling coordinates, as we now demonstrate.

Figure 2 shows a symbolic map of a (static) synthesis of three substructures and its boolean representation. Hypothetically, substructure "A" is represented by flexibilities at coordinate set "A." Substructure "B" is taken to be an unrestrained substructure and so has had grounding springs installed before the calculation of its required flexibilities. These flexibilities have been calculated at coordinate sets "B_c" (connection) and "B_g" (grounding). Flexibilities have been similarly calculated for substructure "C." Substructure "G" represents "ground," a set of coordinates whose associated rows and columns in the relevant matrices would be zeroed out before the synthesis. Substructure "G" is included solely for this exposition; it need not be included in practice, and no flexibilities are required. Coordinate set "A" is (multiply-) connected to coordinate set "B_c"; this connection is denoted "I." The connection response coordinate set associated with the connection "I" consists of coordinate set "A" and coordinate set "B_c." Coordinate set "B_g" is (multiply-) connected to hypothetical coordinate set "G"; this connection is denoted "L," and so forth.



$$[M] = \begin{bmatrix} M_c & 0 \\ 0 & M_g \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \mathrm{I} & \mathrm{J} & \mathrm{K} & \mathrm{L} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right] \\ \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \begin{array}{c} \\ \mathrm{A} \\ \mathrm{B_c} \\ \mathrm{C} \\ \mathrm{B_g} \\ \mathrm{G} \end{array}$$

Fig. 2 A symbolic map and its boolean representation.

Given this symbolic map, we may directly construct all columns of M_c and M_g as follows. All connection response coordinates involved in connection "I" and associated with coordinate set "A" are assigned a "1" in M_c. The corresponding connection response coordinates (those associated with coordinate set "B_c") are assigned a "-1" in M_c. It is arbitrary as to which set carries the "1" or the "-1." The connection response coordinates involved in connection "L" and associated with coordinate set "B_g" are assigned a "1" in M_g. The lowest row of M, row "G" in Fig. 2, is not to be included in an actual synthesis. It is included here to reinforce the concept that the removal of a grounding spring is a special case of an indirect coupling, involving two (connection) response coordinates. In an actual synthesis, we would have zeroed out all rows and columns associated with the pseudosubstructure "G."

E. Structural Synthesis Transformation (SST)

We are now in a position to construct the SST. As discussed earlier the SST operates on Eq. (5) and in effect applies the coupling forces mutually imposed by the various substructures being coupled. The SST transformation matrices are given as Eqs. (17a) and (17b). Note that Eq. (12), modified for the removal of grounding springs, has been integrated into Eq. (17a), where the minus sign indicates that we are considering reaction forces imposed by the grounding spring (its coupling impedance) on the host substructure:

$$\begin{Bmatrix} f_e \\ f_c \\ f_g \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & M_c & 0 \\ 0 & 0 & -M_g\tilde{Z} \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{f}_c \\ \tilde{q}_g \end{Bmatrix} \tag{17a}$$

$$\begin{Bmatrix} q_\sigma \\ q_e \\ \tilde{q}_c \\ \tilde{q}_g \end{Bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & M_c^T & 0 \\ 0 & 0 & 0 & M_g^T \end{bmatrix} \begin{Bmatrix} q_\sigma \\ q_e \\ q_c \\ q_g \end{Bmatrix} \tag{17b}$$

Operating on Eq. (5), these transformations produce,

$$\begin{Bmatrix} q_\sigma \\ q_e \\ \tilde{q}_c \\ \tilde{q}_g \end{Bmatrix} = \begin{bmatrix} Y_{\sigma e} & Y_{\sigma c}M_c & -Y_{\sigma g}M_g\tilde{Z} \\ Y_{ee} & Y_{ec}M_c & -Y_{eg}M_g\tilde{Z} \\ M_c^T Y_{ce} & M_c^T Y_{cc} M_c & -M_c^T Y_{cg} M_g \tilde{Z} \\ M_g^T Y_{ge} & M_g^T Y_{ge} M_g & -M_g^T Y_{gg} M_g \tilde{Z} \end{bmatrix} \begin{Bmatrix} f_e \\ \tilde{f}_c \\ \tilde{q}_g \end{Bmatrix} \tag{18}$$

which is the postsynthesis analog to Eq. (5): a frequency response function model of the synthesized assembly. We extract the third and fourth rows of Eq. (18),

$$\begin{Bmatrix} \tilde{q}_c \\ \tilde{q}_g \end{Bmatrix} = \begin{bmatrix} M_c^T Y_{ce} \\ M_g^T Y_{ge} \end{bmatrix} \{f_e\} + \begin{bmatrix} M_c^T Y_{cc} M_c & -M_c^T Y_{cg} M_g \tilde{Z} \\ M_g^T Y_{ge} M_g & -M_g^T Y_{gg} M_g \tilde{Z} \end{bmatrix} \begin{Bmatrix} \tilde{f}_c \\ \tilde{q}_g \end{Bmatrix} \tag{19}$$

Compatibility is imposed on the connection differential response coordinates,

$$\tilde{q}_c = \mathbf{0}$$

and Eq. (19) for the coupling coordinates,

$$\begin{Bmatrix} \tilde{f}_c \\ \tilde{q}_g \end{Bmatrix} = \begin{bmatrix} -M_c^T Y_{ce} M_c & M_c^T Y_{cg} M_g \tilde{Z} \\ -M_g^T Y_{gc} M_g & I + M_g^T Y_{gg} M_g \tilde{Z} \end{bmatrix}^{-1} \begin{bmatrix} M_c^T Y_{ce} \\ M_g^T Y_{ge} \end{bmatrix} \{f_e\} \tag{20}$$

This equation relates the externally applied generalized forces to the coupling coordinate responses. We now extract the first two rows from Eq. (18), into which Eq. (20) is substituted,

$$\begin{Bmatrix} \tilde{q}_\sigma \\ \tilde{q}_e \end{Bmatrix} = \begin{bmatrix} Y_{\sigma e} \\ Y_{ee} \end{bmatrix} \{f_e\} + \begin{bmatrix} Y_{\sigma c} M_c & -Y_{\sigma g} M_g \tilde{Z} \\ Y_{ec} M_c & -Y_{eg} M_g \tilde{Z} \end{bmatrix} \cdot \begin{bmatrix} -M_c^T Y_{cc} M_c & M_c^T Y_{cg} M_g \tilde{Z} \\ -M_g^T Y_{gc} M_g & I + M_g^T Y_{gg} M_g \tilde{Z} \end{bmatrix}^{-1} \begin{bmatrix} M_c^T Y_{ce} \\ M_g^T Y_{ge} \end{bmatrix} \{f_e\} \tag{21}$$

which leads to the operative form of the SST,

$$\begin{bmatrix} Y_{\sigma e} \\ Y_{ee} \end{bmatrix}^* = \begin{bmatrix} Y_{\sigma e} \\ Y_{ee} \end{bmatrix} - \begin{bmatrix} Y_{\sigma c} & Y_{\sigma g} \\ Y_{ec} & Y_{eg} \end{bmatrix} \begin{bmatrix} M_c & 0 \\ 0 & M_g \end{bmatrix} \cdot \begin{bmatrix} M_c^T Y_{cc} M_c & M_c^T Y_{cg} M_g \\ M_g^T Y_{gc} M_g & \tilde{Z}^{-1} + M_g^T Y_{gg} M_g \end{bmatrix}^{-1} \begin{bmatrix} M_c & 0 \\ 0 & M_g \end{bmatrix}^T \begin{bmatrix} Y_{ce} \\ Y_{ge} \end{bmatrix}^T \tag{22}$$

All frequency response (Y) terms on the right-hand side of Eq. (22) are found from presynthesis substructure frequency response data. The term on the left-hand side is the stress and displacement frequency response function for the synthesized assembly model; the $[\bullet]^*$ denotes a synthesized quantity. Note that all displacement response coordinates are synthesized, being that $e = i \cup c \cup g$. As only the connection and stress response coordinates are required in the assembly stress analysis, computational efficiency is achieved by retaining in the synthesis only the minimal coordinate set, $\sigma \cup c \cup g$.

F. Stress Due to Fastener Tolerance Accumulation

We now develop the relation that is used to directly calculate assembly stress due to the fastener tolerances. Following this, we develop the differential response representation of tolerances required for this calculation. We make use of the synthesized frequency response information provided by Eq. (22). We extract the following relation from the lower partition of the left-hand side of Eq. (22),

$$q_c = Y_{cc}^* f_c \tag{23}$$

Using Eqs. (10a) and (10b), Eq. (23) is transformed into coupling coordinates,

$$\tilde{q}_c = \tilde{Y}_{cc}^* \tilde{f}_c \tag{24}$$

where $\tilde{Y}_{cc}^* = M_c^T Y_{cc}^* M_c$. Equation (24) defines a differential response/coupling force frequency response. The upper partition of the left-hand side of Eq. (23) provides

$$q_\sigma = Y_{\sigma c}^* f_c \tag{25}$$

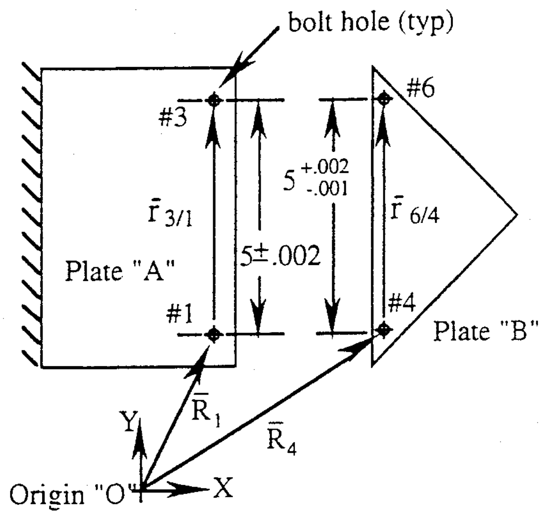


Fig. 3 Assembly of two components.

and we introduce into Eq. (25) the coupling transformation of forces,

$$q_\sigma = \tilde{Y}_{\sigma c}^* f_c \tag{26}$$

Equation (26) defines a stress-coupling force frequency response. Using Eqs. (24) and (26), we obtain the relation required for the calculation of stress, q_σ, due to imposed fastener tolerances:

$$q_\sigma = \tilde{Y}_{\sigma c}^* (\tilde{Y}_{cc}^*)^{-1} \tilde{q}_c \tag{27}$$

where $\tilde{f}_c = (\tilde{Y}_{cc}^*)^{-1} \tilde{q}_c$, found from Eq. (24), has been used.

We focus now on representing the accumulated fastener tolerances as the coupling response coordinate vector $\tilde{q}_c$, for use in Eq. (27). We will refer informally to a "fastener location" as simply a "fastener." We limit our representation to the common situation where, for a given component, all fasteners in that component are located with respect to a specific single "reference" fastener location, also in that component. Note that the fasteners correspond to the nodes on the various component models for which any or all of the response coordinates associated with that node are connection response coordinates. The reader can refer to Fig. 3 during the course of the discussion that follows.

In writing the positions of the various fasteners, we will refer to a global coordinate system that is not associated with a specific component. In addition, for each component we define a local coordinate system, the origin of which is at the reference fastener (labeled "f") for that component. An overbar denotes a vector quantity; an uppercase "R" denotes a vector in the global (component-independent) coordinate system; a lowercase "r" denotes a vector in a component local coordinate system. We write the absolute position $\bar{R}_i$ of the ith fastener location as

$$\bar{R}_i = \bar{R}_f + \bar{r}_{i/f} \tag{28}$$

where $\bar{R}_f$ is the position of the reference fastener (or equivalently, the position of the origin of the component local coordinate system) in the global coordinate system, and $\bar{r}_{i/f}$ is the position of the ith fastener in the local system of the associated component. The quantity $\bar{r}_{i/f}$ is itself comprised of

$$\bar{r}_{i/f} = \bar{r}_{i/f}^{\text{nom}} \pm \Delta \bar{r}_{i/f} \tag{29}$$

where $\bar{r}_{i/f}^{\text{nom}}$ is the nominal position of the ith fastener in the local system, and $\pm\Delta\bar{r}_{i/f}$ is the tolerance specified for the fastener, also in the component local system. In general practice, $|+\Delta\bar{r}_{i/f}| \neq |-\Delta\bar{r}_{i/f}|$. If we transform $\bar{R}_f$ and $\bar{r}_{i/f}$ into a common coordinate system, the absolute position of the ith fastener $\bar{R}_i$ is written in matrix form as

$$\bar{R}_i \equiv \{R_i\} = \begin{Bmatrix} x_i \pm \Delta x_i \\ y_i \pm \Delta y_i \\ z_i \pm \Delta z_i \end{Bmatrix} \tag{30}$$

and similarly the position of the jth fastener is $\{R_j\}$.

If the ith fastener is associated with component "I" and the jth fastener is associated with component "J," we may assemble components "I" and "J" using any or all of the (connection) response coordinates associated with these two fastener locations, i and j. Recognizing that $\bar{R}_f + \bar{r}^{\text{nom}}_{i/f} = \bar{R}_f + \bar{r}^{\text{nom}}_{j/f}$, i.e., the two fasteners occupy the same nominal position in space, and the mismatch between fastener location i and fastener location j is

$$[\mathbf{M}_c]^T \begin{Bmatrix} \boldsymbol{R}_i \\ \boldsymbol{R}_j \end{Bmatrix} = \tilde{q}_c = (\pm\Delta\bar{r}_{i/f})(\pm\Delta\bar{r}_{j/f}) \tag{31}$$

where $[\boldsymbol{M}_c] = [1 \quad -1]^T$, the boolean transformation for a single connection, and $\tilde{q}_c$ is the associated coupling coordinate. It is important to note that $\pm\Delta\bar{r}_{i/f}$ and $\pm\Delta\bar{r}_{j/f}$ must be transformed into a common coordinate system before their combination. Equation (31) is the differential response representation of tolerance required by Eq. (27), the synthesized assembly model.

Figure 3 shows the assembly of two plates, including the vector quantities defined earlier. Plate "A" accommodates two fasteners at locations #1 and #3, as does plate "B" at locations #4 and #6. Plate "A" and plate "B" are to be fastened together by fastening #1 to #4 and #3 to #6. Fastener #1 is taken as the reference fastener, and hence the origin of the local coordinate system for plate "A," as is fastener #4 for plate "B."

G. Combination of Tolerances

It is seen from Eq. (31) that, for a single connection involving the difference of two signed tolerances, four values for the coupling coordinate and four unique stress states are thus defined. For the general case of n tolerances, a thorough accounting of all possible stress states requires the systematic tabulation and application of all tolerance combinations. Such a systematic approach is described later. It may be highly conservative to simply take the value of largest magnitude for a specific tolerance as the worst case, and this precludes the identification of constructively (and destructively) combined stress states. Therefore, as a prerequisite to the tabulation of tolerance combinations, the analyst must be concerned with identifying the number of tolerances involved in a given assembly.

A binary counting is used to tabulate the various combinations of the signed tolerances. For n_t tolerances, we count from 0 to $(n_t)^2 - 1$, in base-2 arithmetic, using a binary word length of n_t. Each binary word is comprised of n_t bits, b_1 through b_{nt}, each of which is either a 0 or a 1. The jth bit in the kth binary word is used to determine the sign of the jth tolerance (and the associated magnitude, if so specified) in the kth tolerance combination. This tabulation is constructed in the numerical example that follows.

IV. Numerical Example of Assembly Stress Analysis

A simple numerical example of assembly stress analysis is presented. The components of Fig. 3 will be assembled, and the static stress due to the specified tolerance accumulation will be calculated. The synthesis (substructure coupling) will be performed using Eq. (23), thus demonstrating the partitioning required. Plane stress finite element models of both components (shown in Fig. 4) provide the required flexibilities. As a static analysis is being performed and substructure "B" is an unrestrained component, three springs to ground, each of rate k_g, have been installed in the model before the calculation of the flexibilities. These springs will be removed simultaneously with the substructure coupling.

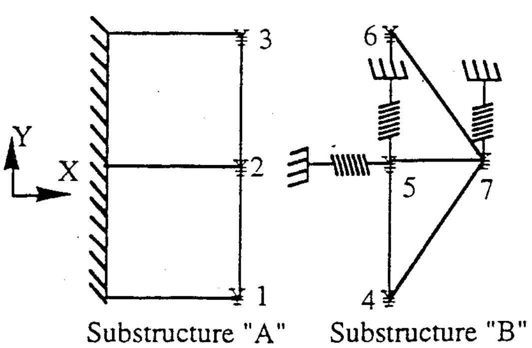


Fig. 4 Finite element models of components.

We choose to synthesize all displacement responses and selected stress responses. Referring to Eq. (23), the bulk of the effort is in constructing $\boldsymbol{Y}_{\sigma e}$ and $\boldsymbol{Y}_{ee}$, as all quantities except for the $\boldsymbol{M}$ and $\boldsymbol{Z}$ matrices are partitions of $\boldsymbol{Y}_{\sigma e}$ and $\boldsymbol{Y}_{ee}$. The required coordinate sets are as follows:

For Substructure "A":

Internal response coordinates $= \{q_{2x}\ q_{2y}\}$

Connection response coordinates $= \{q_{1x}\ q_{1y}\ q_{3x}\ q_{3y}\}$

Stress response coordinates $= \{q^{\sigma}_{2x}\ q^{\sigma}_{2y}\ q^{\tau}_{2xy}\}$

For Substructure "B":

Internal response coordinates $= \{q_{7x}\}$

Connection response coordinates $= \{q_{4x}\ q_{4y}\ q_{6x}\ q_{6y}\}$

Grounding response coordinates $= \{q_{5x}\ q_{5y}\ q_{7y}\}$

Stress response coordinates $= \{q^{\sigma}_{x}\ q^{\sigma}_{y}\ q^{\tau}_{xy}\}$

(stresses associated with lower element)

The coordinate set $e = i^A \cup i^B \cup c^A \cup c^B \cup g^B$

$= \{q_{2x}\ q_{2y}\ q_{7x}\ q_{1x}\ q_{1y}\ q_{3x}\ q_{3y}\ q_{4x}\ q_{4y}\ q_{6x} q_{6y} q_{5x} q_{5y} q_{7y}\}$

The coordinate set $\sigma = \{q^{\sigma}_{2x}\ q^{\sigma}_{2y}\ q^{\gamma}_{2xy}\ q^{\sigma}_{x}\ q^{\sigma}_{y} q^{\gamma}_{xy}\}$

To generate the flexibilities (frequency response) defined by these sets, static unit loads are applied at the coordinates defined by the set $\{e\}$. To establish the coupling between substructures and to remove the grounding springs in substructure "B," the boolean mapping matrices are constructed from the knowledge of the connectivity. The columns of the mapping matrices each correspond to a connection to be established between connection response coordinates:

$$\boldsymbol{M}_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{matrix} 1x \\ 1y \\ 3x \\ 3y \\ 4x \\ 4y \\ 6x \\ 6y \end{matrix} \quad \begin{matrix} \\ \\ \text{Column 1: } 1x \text{ to } 4x \\ \text{Column 2: } 1y \text{ to } 4y \\ \text{Coulmn 3: } 3x \text{ to } 6x \\ \text{Coulmn 4: } 3y \text{ to } 6y \\ \\ \\ \end{matrix}$$

$$\boldsymbol{M}_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} 5x \\ 5y \\ 7y \end{matrix} \quad \begin{matrix} \text{Column 1: } 5x \text{ to ground} \\ \text{Column 2: } 5y \text{ to ground} \\ \text{Coulmn 3: } 7y \text{ to ground} \end{matrix}$$

The associated coupling impedance matrix $\tilde{\boldsymbol{Z}}$ that contains the grounding spring rates can be similarly constructed as $\tilde{\boldsymbol{Z}} = \text{diag}(k_g^{-1}, k_g^{-1}, k_g^{-1})$.

Once the matrix operations required by Eq. (23) have been performed, the partitions of $\lfloor Y_{e\sigma} Y_{ee} \rfloor^T$ relevant to the calculation of tolerance stress, Eq. (27), are extracted.

What remains to be identified is the set of tolerances comprising $\tilde{q}_c$ that are to be applied to the synthesized assembly model, Eq. (27). Consider the previous example involving bolted connections #1 to #4 and #3 to #6. The coupling response vector is

$$\{\tilde{q}_c\} = \begin{Bmatrix} \tilde{q}_{1-4} \\ \tilde{q}_{3-6} \end{Bmatrix} = \begin{Bmatrix} 0 \\ (\pm\Delta_{\bar{r}3/1} - \pm\Delta_{\bar{r}6/4}) \end{Bmatrix}$$

As two tolerances are involved (specified in Fig. 3), we construct the binary sequence required by counting from 0 to $(n_t)^2 - 1 = 3$. This is shown in Table 1.

The $\tilde{q}_c$ vectors corresponding to the four tolerance combinations in Table 1 are therefore

$$\{\tilde{q}_c\}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \{\tilde{q}_c\}^{(2)} = \begin{Bmatrix} 0 \\ 0.003 \end{Bmatrix}$$

$$\{\tilde{q}_c\}^{(3)} = \begin{Bmatrix} 0 \\ -0.004 \end{Bmatrix} \qquad \{\tilde{q}_c\}^{(4)} = \begin{Bmatrix} 0 \\ -0.003 \end{Bmatrix}$$

Table 1 Tolerance combinations

Decimal	Binary		Sign		Combined value
no.	b_1	b_2	$\Delta\bar{r}_{3/1}$	$\Delta\bar{r}_{6/4}$	$\bar{q}_{3-6}$
0	0	0	+	+	(+0.002) − (+0.002)
1	0	1	+	−	(+0.002) − (−0.001)
2	1	0	−	+	(−0.002) − (+0.002)
3	1	1	−	−	(−0.002) − (−0.001)

These vectors are substituted successively into Eq. (27) for the direct calculation of the stress due to tolerance accumulation.

V. Summary

The manufacture of components for subsequent assembly generally requires that the location of fasteners be specified with allowable deviations from nominal location, known as tolerance. When such components are assembled, the mismatch in fastener location, or tolerance accumulation, can generate stresses. A method for the efficient analysis of this assembly stress has been presented. The models of the components to be assembled are frequency response function models, and their assembly is performed using frequency domain structural synthesis. The synthesis is carried out using a set of differential response coordinates as they inherently organize the establishment of complex connectivities in synthesis and also allow the direct and convenient application of the fastener tolerances to the synthesized assembly model. The frequency domain structural representation inherently provides a theoretically exact and dimensionally unlimited reduction in the assembled system model. Furthermore, the inclusion of stress and strain coordinate frequency response allows the rapid calculation of said quantities from the assembled system model. What results from the method is a low-order model of the assembled components that can be used for efficient calculation of assembly stress due to the accumulation of fastener tolerances. This allows a designer to specify maximum tolerances, hence reducing manufacturing costs and minimizing manufacturing time.

References

[1]Gordis, J. H., Bielawa, R. L., and Flannelly, W. G., "A General Theory for Frequency Domain Structural Synthesis," *Journal of Sound and Vibration*, Vol. 150, No. 1, 1989, pp. 139–158.

[2]Gordis, J. H., "On Structural Synthesis and Identification in the Frequency Domain," Doctoral Thesis, Rensselaer Polytechnic Inst., Troy, NY, Aug. 1990.

[3]Jetmundsen, B., Bielawa, R. L., and Flannelly, W. G., "Generalized Frequency Domain Substructure Synthesis," *Journal of the American Helicopter Society*, Vol. 33, No. 1, 1988, pp. 55–64.

[4]Jetmundsen, B., "On Frequency Domain Methodologies for Prescribed Structural Modification and Subsystem Synthesis," Doctoral Thesis, Rensselaer Polytechnic Inst., Troy, NY, 1986.